Mathematical Approaches in GNSS Positioning and Integrity Monitoring

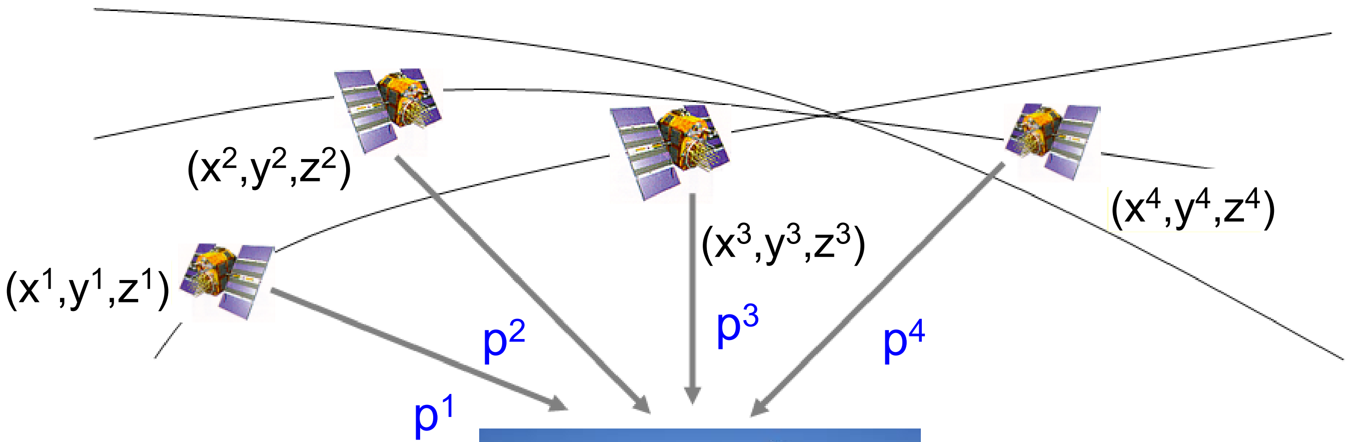
David Minkwitz
German Aerospace Center (DLR), Institute of Communications and Navigation

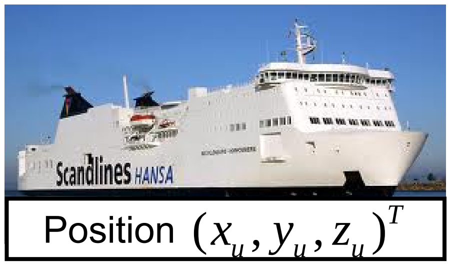
DLR

Outline

- General Principle of Satellite Navigation
- Real-Time Kinematic
  - Double Differenced Measurement Model
  - Kalman Filter Approach for Float Solutions
  - Integer Least Squares Approach for Fixed Solutions
- Issues of Integrity Monitoring
  - Ambiguity Validation
  - Fault Detection, Exclusion and Protection Level

Principle of Satellite Navigation – Positioning





4 Unknows

Receiver Position (x_u, y_u, z_u) *and*

Receiver Clock Error dt_u

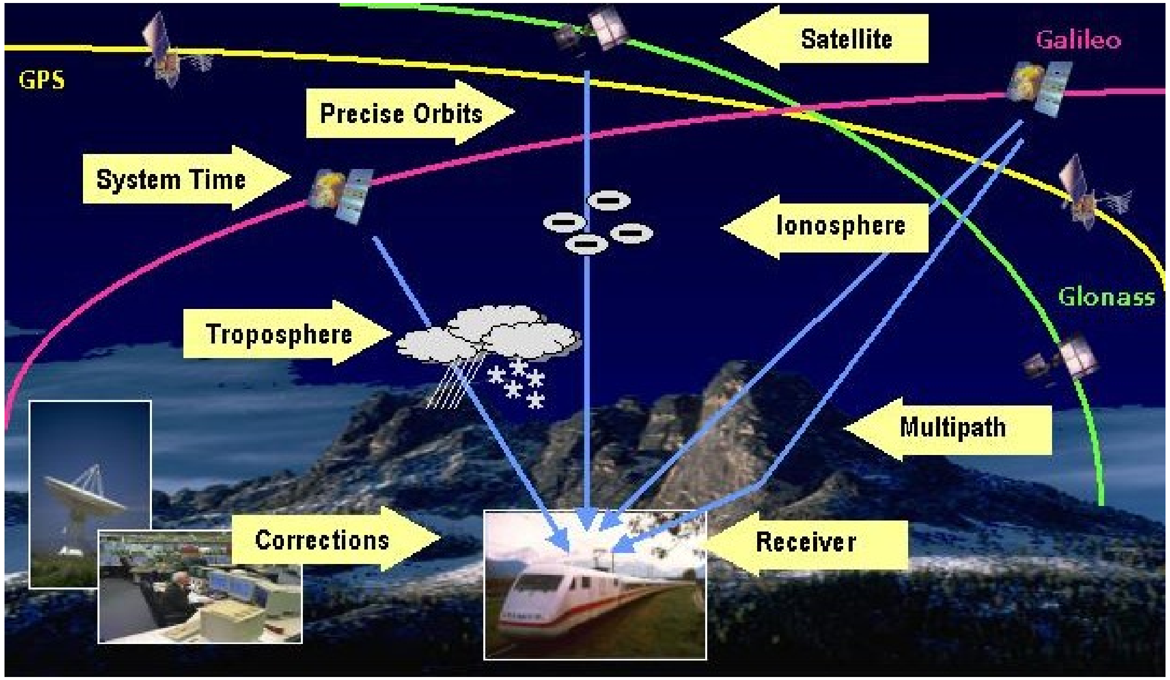
Search for Estimation of

$$\mathbf{x} = (x_u \quad y_u \quad z_u \quad dt_u)^T$$

Simplified Measurement Model for Code Observations

$$p^j = \underbrace{\sqrt{(x^j - x_u)^2 + (y^j - y_u)^2 + (z^j - z_u)^2}}_{=: \text{ geometric range } r^j} + c \cdot dt_u + \varepsilon_p^j$$

Principle of Satellite Navigation – Error Sources



Common Measurement Model for Code and Phase Observations considering GNSS Error Sources

$$p^j = r^j + c \cdot (dt_u - dt^j) + I^j + T^j + \varepsilon_p^j$$

$$\Phi^j = r^j + c\left(\delta t_u - \delta t^j\right) + \lambda N^j - I^j + T^j + \varepsilon_\Phi^j \quad \text{with} \quad \varepsilon_\Phi^j << \varepsilon_p^j$$

[Misra-2006]

Principle of Satellite Navigation – Single Point Positioning (SPP)

Given: Code Measurements $p^j,\ j \in \{1,\ldots,n\}$,

Satellite Positions $\mathbf{x}^j = (x^j \quad y^j \quad z^j)^T$

Unknown: Position and Receiver Clock Error $\mathbf{x} = (x_u \quad y_u \quad z_u \quad dt_u)^T$

$$p^j = \sqrt{(x_u - x^j)^2 + (y_u - y^j)^2 + (z_u - z^j)^2} + dt_u \cdot c + \varepsilon_p^j, \text{ speed of light } c$$

Linearization by First Order Taylor Series in Point $\mathbf{x}_0$ [Blewitt-1997]

$$p^j(\mathbf{x}) \approx p^j(\mathbf{x}_0) + \dot{p}^j(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

$$= p^j(\mathbf{x}_0) + \frac{\partial p^j}{\partial x_u}\Big|_{\mathbf{x}_0} \underbrace{(x_u - x_{u,0})}_{=:\Delta x_u} + \frac{\partial p^j}{\partial y_u}\Big|_{\mathbf{x}_0} \underbrace{(y_u - y_{u,0})}_{=:\Delta y_u} + \frac{\partial p^j}{\partial z_u}\Big|_{\mathbf{x}_0} \underbrace{(z_u - z_{u,0})}_{=:\Delta z_u} + \frac{\partial p^j}{\partial t_u}\Big|_{\mathbf{x}_0} \underbrace{(dt_u - dt_{u,0})}_{=:\Delta dt_u}$$

Principle of Satellite Navigation – Positioning

Set of n linear equations

$$\Delta\mathbf{p} = \overbrace{\begin{pmatrix} p^1 - p^1(\mathbf{x}_0) \\ p^2 - p^2(\mathbf{x}_0) \\ \dots \\ p^n - p^n(\mathbf{x}_0) \end{pmatrix}}^{p^{\text{observed}} - p^{\text{computed}}} = \begin{bmatrix} \frac{\partial p^1}{\partial x} & \frac{\partial p^1}{\partial y} & \frac{\partial p^1}{\partial z} & \frac{\partial p^1}{\partial dt} \\ \frac{\partial p^2}{\partial x} & \frac{\partial p^2}{\partial y} & \frac{\partial p^2}{\partial z} & \frac{\partial p^2}{\partial dt} \\ \dots & \dots & \dots & \dots \\ \frac{\partial p^n}{\partial x} & \frac{\partial p^n}{\partial y} & \frac{\partial p^n}{\partial z} & \frac{\partial p^n}{\partial dt} \end{bmatrix} \begin{pmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ \Delta dt_u \end{pmatrix} + \begin{pmatrix} \varepsilon_p^1 \\ \varepsilon_p^2 \\ \dots \\ \varepsilon_p^n \end{pmatrix} = A\Delta\mathbf{x} + \boldsymbol{\varepsilon}_p$$

- in the most cases over-determined system of equations, but ill-posed
- ill-conditioned $A \in \mathbb{R}^{n\times 4}$
- assumption: $\varepsilon_p^j \sim N(0,\sigma^2),\ j \in \{1,\dots,n\}$

Principle of Satellite Navigation – Positioning

Goal: Estimation of $\Delta \mathbf{x}$

Least Squares / Gauß-Markow Approach $\min_{\Delta \mathbf{x} \in \mathbb{R}^{4\times 1}} || \Delta \mathbf{p} - A \Delta \mathbf{x} ||^2$

$$\Delta \mathbf{p} = A \Delta \mathbf{x} + \boldsymbol{\varepsilon}_p \quad \Leftrightarrow \quad A^T \Delta \mathbf{p} = A^T A \Delta \mathbf{x}$$

$$\Leftrightarrow \quad \Delta \hat{\mathbf{x}} = (A^T A)^{-1} A^T \Delta \mathbf{p}$$

Iterative Calculation

Step 1: start position $\mathbf{x}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T$, calculate A and $\Delta \mathbf{p}$

$\longrightarrow$ calculate $\Delta \hat{\mathbf{x}}$

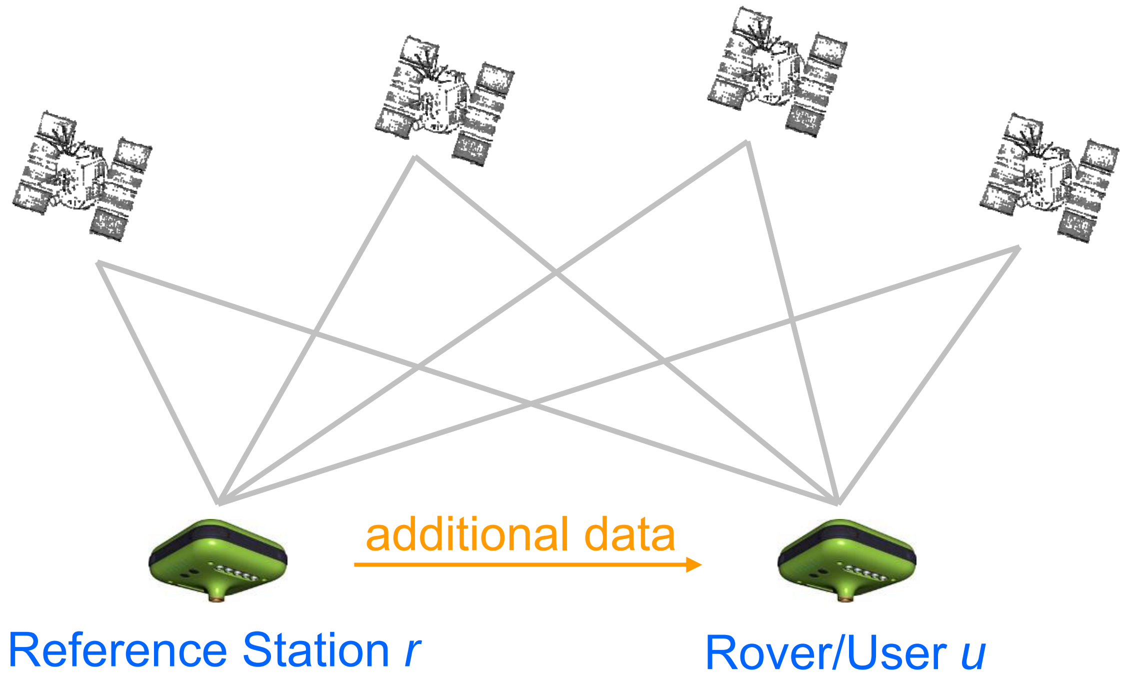
$\longrightarrow$ $\mathbf{x}_1 = \mathbf{x}_0 + \Delta \hat{\mathbf{x}}$

Step 2: position $\mathbf{x}_1$ calculate new A and $\Delta \mathbf{p}$ $\longrightarrow$ calculate $\Delta \hat{\mathbf{x}}$ $\longrightarrow$ $\mathbf{x}_2 = \mathbf{x}_1 + \Delta \hat{\mathbf{x}}$

...

Termination Criteria: maximum Iteration (e.g. 10) or $\Delta \hat{\mathbf{x}} \le 10^{-3}$

Ground Based Augmentation System
Real-Time Kinematic (RTK) Algorithm



- Installation of a reference station close to the rover *enables the application of a Real-Time Kinematic (RTK)* algorithm and hence crucial mitigation of error sources like atmosperhic delay and receiver clock error
- Usage of carrier phase measurements instead of pseudorange/code measurements

Real-Time Kinematic (RTK) – General Functionality

GNSS Observations of Reference Station and User

Determination of SPP Solution
First Estimation of baseline vector/user position

Double Difference Calculations
Mitigation of common error sources (e.g. atmosphere, satellite clock errors)

Prediction of $\tilde{\mathbf{x}}_{k+1}^{-}$ $\nabla\Delta\Phi$

Determination of Float Solution with Kalman-Filter

$$\tilde{\mathbf{x}}_k^+ = \left(x_u \quad y_u \quad z_u \; \nabla\Delta N_{ur}^{21} \quad \nabla\Delta N_{ur}^{31} \quad \dots \quad \nabla\Delta N_{ur}^{[(n-1)\cdot\#\, freq+1]1}\right)^T$$

$\tilde{\mathbf{N}}, Q_{\tilde{\mathbf{N}}}$

Determination of Fixed Solution with ILS
Final Estimation of baseline vector/user position

Double Differenced Measurement Model

Carrier phase measurements of satellite *j* on frequency *i* at reference *r* & user *u*

$$\Phi_{i,u}^{j} = r_u^j + c\left(\delta t_u - \delta t^j\right) + \lambda_i N_u^j - I_{i,u}^j + T_u^j + \varepsilon_u^j$$

$$\Phi_{i,r}^{j} = r_r^j + c\left(\delta t_r - \delta t^j\right) + \lambda_i N_r^j - I_{i,r}^j + T_r^j + \varepsilon_r^j$$

Construction of **Single Differences**

$$\Delta\Phi_{i,ur}^{j} = \Phi_{i,u}^{j} - \Phi_{i,r}^{j} = \Delta r_{ur}^{j} + c(\delta t_u - \delta t_r) - \Delta I_{i,ur}^{j} + \Delta T_{ur}^{j} + \lambda_i \Delta N_{ur}^{j} + \Delta\varepsilon_{\Phi,ur}^{j}$$

→ Elimination of satellite clock error & mitigation of atmosphere errors

Construction of **Double Differences**

$$\nabla\Delta\Phi_{i,ur}^{jl} = \Delta\Phi_{i,ur}^{j} - \Delta\Phi_{i,ur}^{l} = \nabla\Delta r_{ur}^{jl} \underbrace{- \nabla\Delta I_{i,ur}^{jl} + \nabla\Delta T_{ur}^{jl}}_{\text{for short baselines} \approx 0} + \lambda_i \nabla\Delta N_{ur}^{jl} + \nabla\Delta\varepsilon_{\Phi,ur}^{jl}$$

→ Elimination of receiver clock error & further mitigation of atmosphere errors

Linarization of Double Differenced Measurement Model

Underdetermined system of $M := (n-1)\cdot\ \#freq$ **non-linear** equations with ***M+3*** **unknowns**

$$\nabla\Delta\Phi_{ur}^{jl} = \nabla\Delta r_{ur}^{jl} + \lambda_i \nabla\Delta N_{ur}^{jl} + \nabla\Delta\varepsilon_{\Phi,ur}^{jl}$$

Linearization by First Order Taylor Series

$$\Delta\mathbf{x}_u = \Delta\mathbf{x}_{ur} := \Delta\mathbf{x}_u - \Delta\mathbf{x}_r$$

$$\Delta\mathbf{x}_r = 0, \text{ because } \mathbf{x}_r \text{ exactly known}$$

$$\underbrace{\begin{pmatrix} \nabla\Delta\Phi_{ur}^{21} - \nabla\Delta\Phi_{ur}^{21,computed} \\ \nabla\Delta\Phi_{ur}^{31} - \nabla\Delta\Phi_{ur}^{31,computed} \\ \cdots \\ \nabla\Delta\Phi_{ur}^{(M+1)1} - \nabla\Delta\Phi_{ur}^{(M+1)1,computed} \end{pmatrix}}_{=:\mathbf{z}_k} = \begin{pmatrix} \frac{\partial\nabla\Delta\Phi_{ur}^{21}}{\partial x_u} & \frac{\partial\nabla\Delta\Phi_{ur}^{21}}{\partial y_u} & \frac{\partial\nabla\Delta\Phi_{ur}^{21}}{\partial z_u} \\ \frac{\partial\nabla\Delta\Phi_{ur}^{31}}{\partial x_u} & \frac{\partial\nabla\Delta\Phi_{ur}^{31}}{\partial y_u} & \frac{\partial\nabla\Delta\Phi_{ur}^{31}}{\partial z_u} \\ \cdots & \cdots & \cdots \\ \frac{\partial\nabla\Delta\Phi_{ur}^{(M+1)1}}{\partial x_u} & \frac{\partial\nabla\Delta\Phi_{ur}^{(M+1)1}}{\partial y_u} & \frac{\partial\nabla\Delta\Phi_{ur}^{(M+1)1}}{\partial z_u} \end{pmatrix} \Delta\mathbf{x}_u + \begin{pmatrix} \lambda_i & 0 & \cdots & 0 \\ 0 & \lambda_i & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix} \nabla\Delta\mathbf{N}_{ur} + \nabla\Delta\boldsymbol{\varepsilon}_{\Phi,ur}$$

Kalman Filter Approach for optimal estimation of float solution

$$\hat{\mathbf{x}}_k^{+} = \left(x_u \quad y_u \quad z_u \; \nabla\Delta N_{ur}^{21} \quad \nabla\Delta N_{ur}^{31} \quad \ldots \quad \nabla\Delta N_{ur}^{(M+1)1} \right)^T$$

$$\longrightarrow \quad \hat{\mathbf{x}}_k^{+} = \hat{\mathbf{x}}_k^{-} + K_k(\mathbf{z}_k - H_k \Delta\hat{\mathbf{x}}_k^{-}), \qquad K_k = P_k^{-} H_k^T (H_k P_k^{-} H_k^T + R_k)^{-1}$$

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Integer Least Squares Approach for Fixed Solution

Approach: $\min \| z - G\Delta\mathbf{x} + A\nabla\Delta\tilde{\mathbf{N}} \|^2,\ \Delta\mathbf{x} \in \mathbb{R}^{3\times1}, \nabla\Delta\tilde{\mathbf{N}} \in \mathbb{R}^{M\times1}$ neglects integer property of ambiguities

→ **new Approach:** $\min \| z - G\Delta\mathbf{x} + A\nabla\Delta\mathbf{N} \|^2,\ \Delta\mathbf{x} \in \mathbb{R}^{3\times1}, \nabla\Delta\mathbf{N} \in \mathbb{Z}^{M\times1}$

Commonly used algorithms in practise:

- Integer rounding
- Integer bootstrapping
- Integer least squares (e.g.**L**east-squares **AMB**iguity **D**ecorrelation **A**djustment)

LAMBDA Idea

$$\tilde{\mathbf{N}} \longrightarrow \boxed{\text{LAMBDA}} \longrightarrow \mathbf{N} \longrightarrow \hat{\mathbf{x}}_k^{fixed} = \begin{pmatrix} x_u^{fixed} \\ y_u^{fixed} \\ z_u^{fixed} \end{pmatrix} = \underbrace{\begin{pmatrix} x_{k,u}^{+} \\ y_{k,u}^{+} \\ z_{k,u}^{+} \end{pmatrix}}_{\hat{\mathbf{x}}_{k,}^{+}(1:3)} - \underbrace{Q_{\hat{\mathbf{x}}_k^+\tilde{\mathbf{N}}}}_{\in\mathbb{R}^{3\times M}} \underbrace{Q_{\tilde{\mathbf{N}}}^{-1}}_{\in\mathbb{R}^{M\times M}} (\tilde{\mathbf{N}} - \mathbf{N})$$

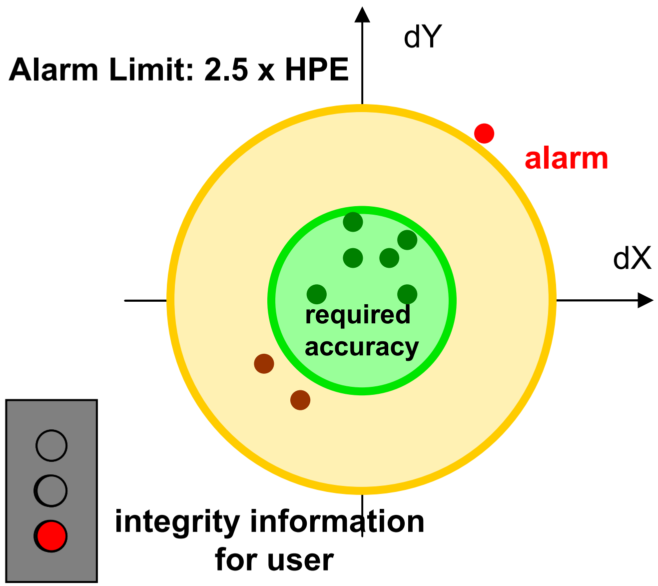
$$\min_{\mathbf{N}\in\mathbb{Z}^{M\times1}} \| \tilde{\mathbf{N}} - \mathbf{N} \|^2_{Q_{\tilde{\mathbf{N}}}^{-1}} = \min_{\mathbf{N}\in\mathbb{Z}^{M\times1}} (\tilde{\mathbf{N}} - \mathbf{N})^T Q_{\tilde{\mathbf{N}}}^{-1} (\tilde{\mathbf{N}} - \mathbf{N}),$$

$$Q_{\tilde{\mathbf{N}}} = P_k^{+}(4:M+3, 4:M+3)$$

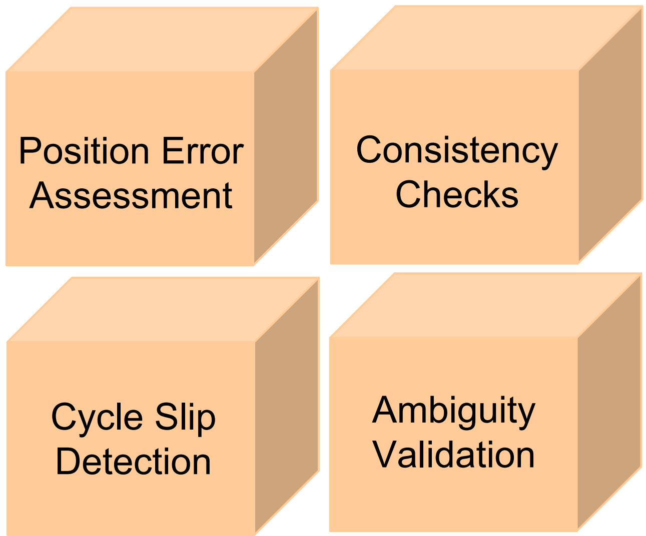
[Teunissen-1998]

Integrity Monitoring

Integrity. The ability to provide users with warnings within a specified time when the system should not be used for navigation. [IMO-A.915]



Holistic Concept for Carrier-Phase Receiver Autonomous Integrity Monitoring (CRAIM)



[Milner-2011]

Real-Time Kinematic (RTK) Module – Currently CRAIM Functionalities within used RTK

Determination of SPP Solution

Consistency Check

Double Difference Calculations

Cycle Slip Detection

$$SD_{\text{geometry Free}}\left(\Phi_{t_i}\right) - SD_{\text{geometryFree}}\left(\Phi_{t_{i-1}}\right) > Threshold\,?$$

Determination of Float Solution with Kalman-Filter

Consistency Check

Determination of Fixed Solution with ILS

Ambiguity Validation by Ratio Test

Currently Missing: Implementation of Position Error Assessment!

Ambiguity Validation – Principle

$$R = (\tilde{\mathbf{N}} - \mathbf{N})^T Q_{\tilde{\mathbf{N}}}^{-1} (\tilde{\mathbf{N}} - \mathbf{N}),\ \tilde{\mathbf{N}}\text{: float ambiguities vector}\ \ \mathbf{N}\text{: integer ambiguities vector}$$

R_1 : residual for optimal ambiguity vector

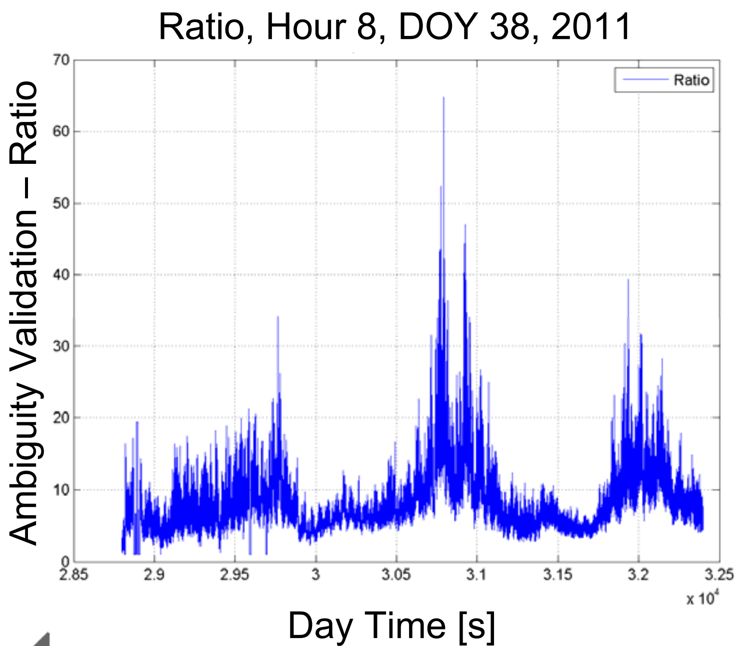
R_2 : residual for second optimal ambiguity vector

Ratio $\dfrac{R_2}{R_1} > T$?

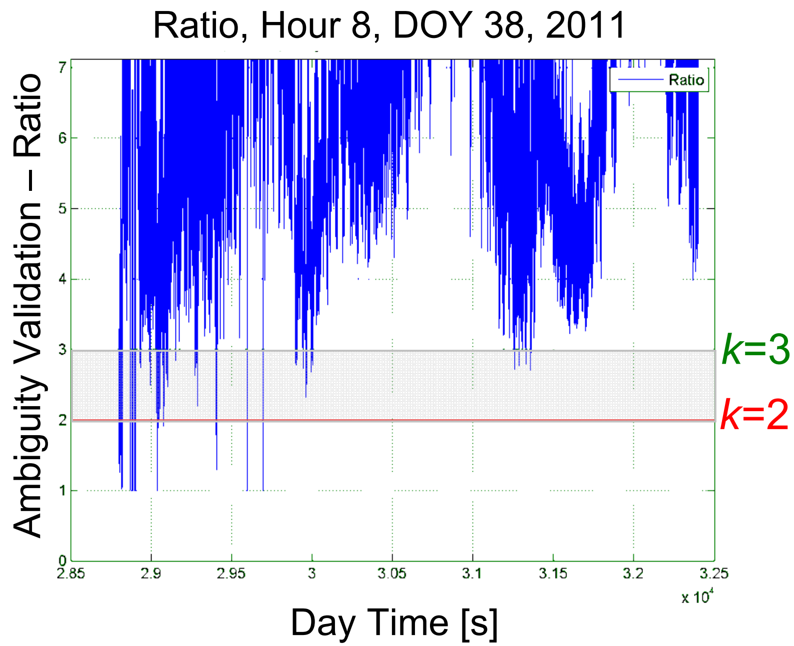
→ yes: take fixed solution

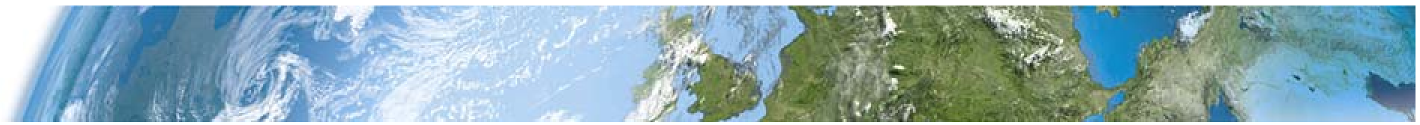
→ no: take float solution

T currently fixed emperical value between 2 and 3 ← No mathematical justification



Zoom in *10^{-1}

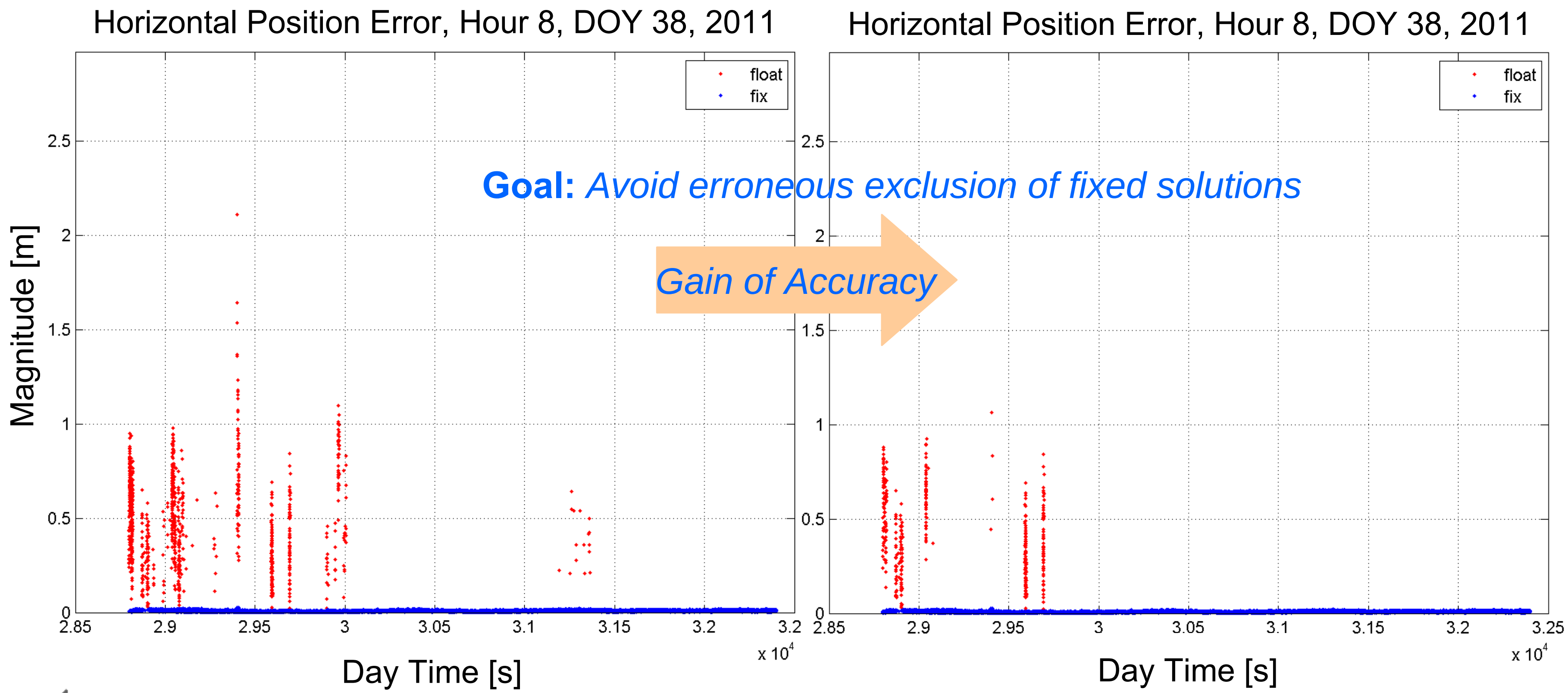




Ambiguity Validation – Issue Example

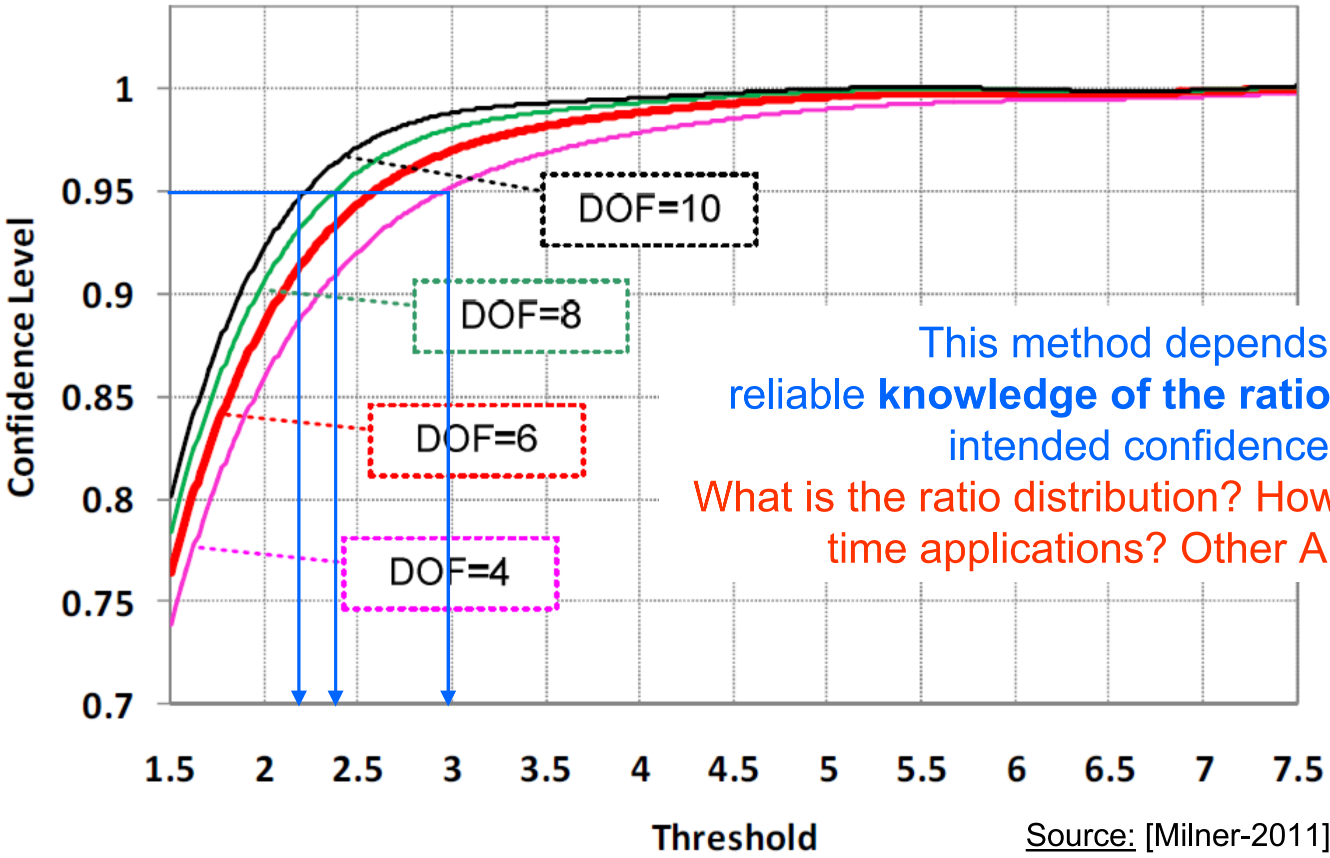
Processing 1 $\frac{R_2}{R_1} > 3?$

Processing 2 $\frac{R_2}{R_1} > 2?$



Ambiguity Validation – Possible Option for Threshold Determination

Variable Threshold Ambiguity Validation



This method depends on the reliable **knowledge of the ratio distribution** and intended confidence level.

What is the ratio distribution? How determine in real-time applications? Other Approaches?

Source: [Milner-2011]

Consistency Checks

Main Issue:

Optimisation between sensitivity and signal availability: correct fault detection and exclusion

Classical Approach [Borre-2009]

Detection

$$\mathbf{y} = A\Delta\mathbf{x} + \boldsymbol{\varepsilon} \rightarrow \text{Chi-Quadrat-Test} \ \frac{\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}}{\sigma_{\mathbf{y}}^2} \sim \chi_m^2$$

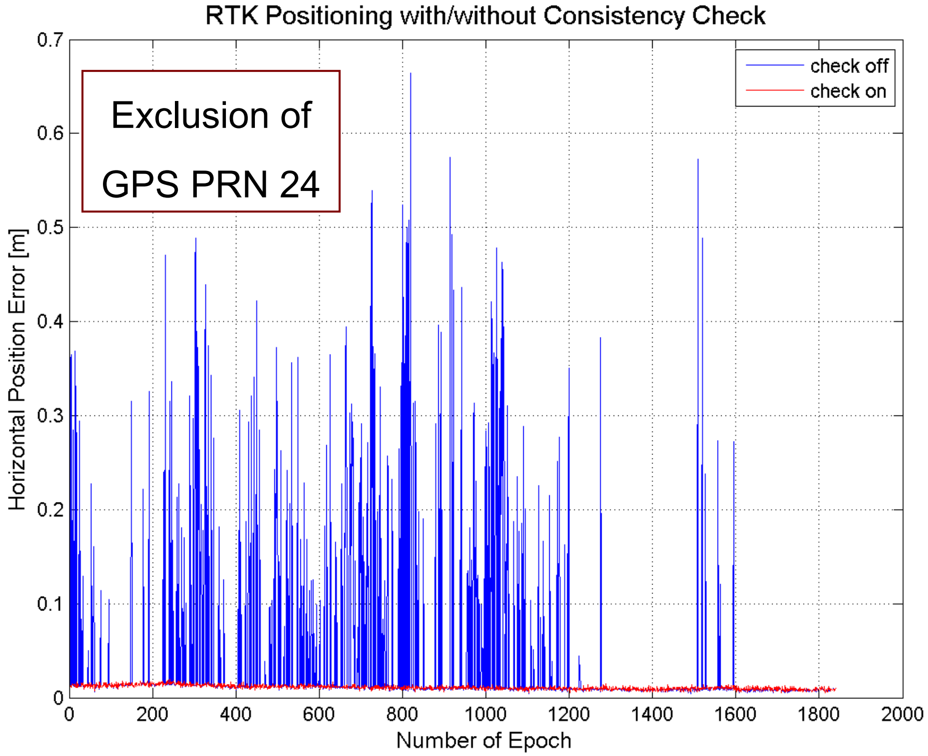
$$\text{Confidence Level} = P(X \leq T) = \int_0^T f_{\chi_m^2}(x)dx \rightarrow \frac{\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}}{\sigma_y^2} > T\ ?$$

Exclusion

Construct n substes with n-1 satellites, calculate test statistic for each subset: exclusion of faulty satellite probably causes the lowest test statistic

Alternative/Advanced Approach [Ene-2009]

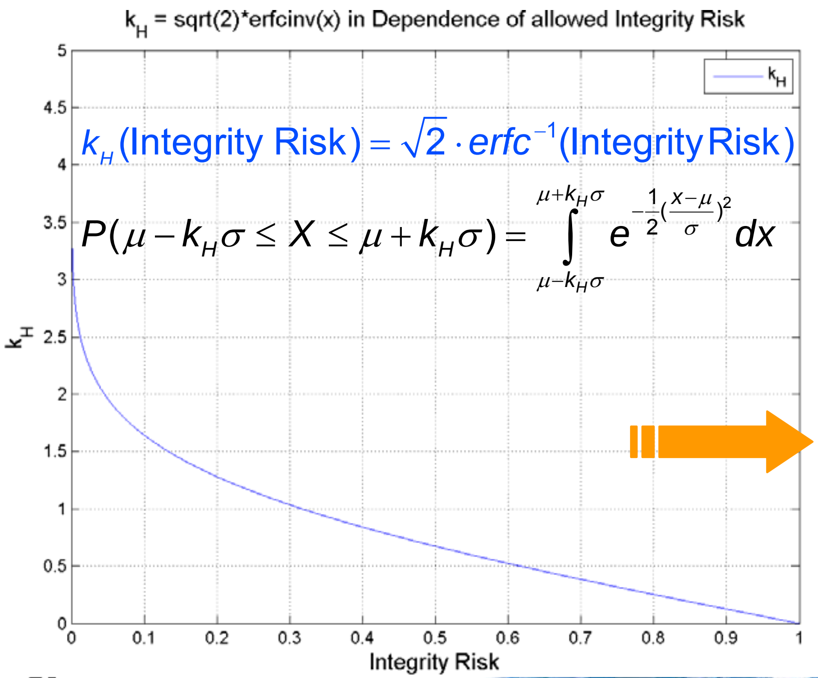
Multiple Hypothesis Solution Separation



Position Error Assessment

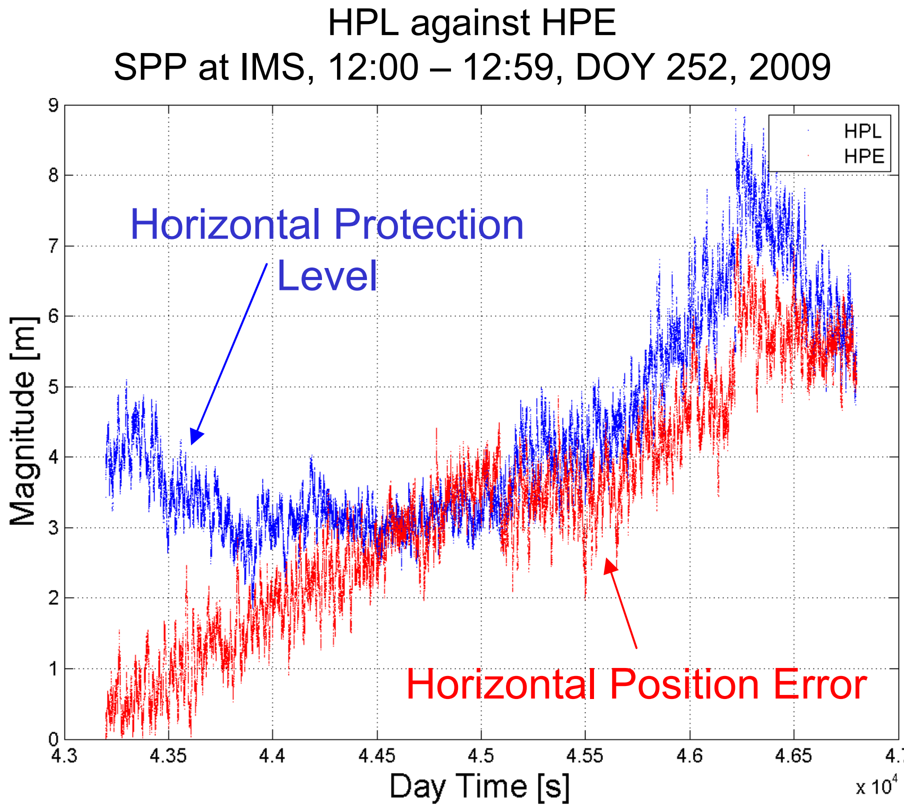
- **Intention:** reliable *overbounding of actual position error* at user site to guarantee integrity without harm of continuity due to too conservative estimations

- Popular approach [Feng-2008]:
  Reminder: $\varepsilon \sim N(0,\sigma^2)$



$$k_H(\text{Integrity Risk}) = \sqrt{2}\cdot erfc^{-1}(\text{Integrity Risk})$$

$$P(\mu - k_H\sigma \le X \le \mu + k_H\sigma) = \int_{\mu-k_H\sigma}^{\mu+k_H\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$



Upper Bound of Horizontal Position Error with maximum allowed Integrity Risk (IR) $\mu + k_H(IR)\cdot\sigma_H$

HPL $= k_H \cdot \sigma_H,\ \sigma_H = \sqrt{P_k^+(1,1) + P_k^+(2,2)}$ and $\mu = 0$

Summary

- Least-squares approaches are applied to calculate position by pseudorange measurements or carrier phase measurements

- Presented approaches differ in the used a priori information, number of unknowns and performance

- Integrity monitoring functionalities facilitate the exclusion of inconsistent measurements and the warranty of reliability

- Challenging Tasks:
    - Increase of position accuracy ***and*** integrity
    - Optimal tuning of covariance matrices
    - Determination of optimal dynamic models within Kalman Filter
    - Reliable ambiguity validation
    - Optimisation between sensitivity and availability during consistency checks
    - Reliable position error assessment
    - Increase of algorithm robustness against measurement errors

References

[Borre-2009]	Borre, K.: *GPS EASY Suite II: easy13 – RAIM, InsideGNSS, July/August, 2009.*
[Brown-1992]	Brown, R. G., Hwang, P. Y.C.: *Introduction to Random Signals and Applied Kalman-Filtering, 2nd Edition, John Wiley & Sons, Inc. , 1992.*
[Chang-2005]	Chang, X.-W., Yang, X., Zhou, T.: *MLAMBDA: A modified LAMBDA method for integer least-squares estimation, McGill University, Quebec, Canada, 2005.*
[Ene-2009]	Ene, A.: *Utilization Of Modernized Global Navigation Satellite Systems For Aircraft-Based Navigation Integrity, Doctor Thesis, Stanford University, 2009.*
[FAA-2011]	Federal Aviation Administration (FAA): *Global Positioning System (GPS) Standard Positioning Service (SPS) Performance Analysis Report, October, 2011.*

References

[Feng-2008]	Feng, S., Ochieng, W., Moore, T., Hill, C., Hide, C.: *Carrier phase-based integrity monitoring fpr high-accuracy positioning, GPS Solution, Springer Press, 2008.*
[Milner-2011]	Milner, C., Feng, S., Jokinen, A., Ochieng, W.: Holistic *Approach to Carrier-Phase Receiver Autonomous Integrity Monitoring (CRAIM), 2011.*
[Misra-2006]	Misra, P., Enge, P.: *Global Positioning System – Signals, Measurements, and Performance, 2nd Edition, Ganga-Jamuna Press, 2006.*
[SDCM]	Russian System of Differential Correction and Monitoring: http://www.sdcm.ru/smglo/staticpages?version=eng&site=extern&title=about
[Teunissen-1998]	Teunissen, P. J. G.; Kleusberg, A.: *GPS for Geodesy, 2nd Edition, Springer Press, 1998.*